BAKER'S EXPLICIT ABC-CONJECTURE AND APPLICATIONS

SHANTA LAISHRAM AND T. N. SHOREY

Dedicated to Professor Andrzej Schinzel on his 70th Birthday

ABSTRACT. The conjecture of Masser-Oesterlé, popularly known as *abc*-conjecture have many consequences. We use an explicit version due to Baker to solve a number of conjectures.

1. Introduction

The well known conjecture of Masser-Oesterle states that

Conjecture 1.1. Oesterlé and Masser's abc-conjecture: *For any given $\epsilon > 0$ there exists a computable constant $\mathfrak{c}_\epsilon$ depending only on ϵ such that if*

$$a + b = c \tag{1}$$

where a, b and c are coprime positive integers, then

$$c \le \mathfrak{c}_\epsilon \left(\prod_{p|abc} p \right)^{1+\epsilon}.$$

It is known as *abc*-conjecture; the name derives from the usage of letters a, b, c in (1). For any positive integer $i > 1$, let $N = N(i) = \prod_{p|i} p$ be the radical of i, $P(i)$ be the greatest prime factor of i and $\omega(i)$ be the number of distinct prime factors of i and we put $N(1) = 1, P(1) = 1$ and $\omega(1) = 0$. An explicit version of this conjecture due to Baker [Bak94] is the following:

Conjecture 1.2. Explicit abc-conjecture: *Let a, b and c be pairwise coprime positive integers satisfying* (1). *Then*

$$c < \frac{6}{5} N \frac{(\log N)^\omega}{\omega!}$$

where $N = N(abc)$ and $\omega = \omega(N)$.

We observe that $N = N(abc) \ge 2$ whenever a, b, c satisfy (1). We shall refer to Conjecture 1.1 as *abc−conjecture* and Conjecture 1.2 as *explicit abc−conjecture.* Conjecture 1.2 implies the following explicit version of Conjecture 1.1.

Key words and phrases. ABC Conjecture, Generalized Fermat Equation.

Theorem 1. *Assume Conjecture 1.2. Let a, b and c be pairwise coprime positive integers satisfying* (1) *and $N = N(abc)$. Then we have*

$$c < N^{1+\frac{3}{4}}. \tag{2}$$

Further for $0 < \epsilon \le \frac{3}{4}$, there exists ω_ϵ depending only ϵ such that when $N = N(abc) \ge N_\epsilon = \prod_{p \le p_{\omega_\epsilon}} p$, we have

$$c < \kappa_\epsilon N^{1+\epsilon}$$

where

$$\kappa_\epsilon = \frac{6}{5\sqrt{2\pi \max(\omega, \omega_\epsilon)}} \le \frac{6}{5\sqrt{2\pi\omega_\epsilon}}$$

with $\omega = \omega(N)$. Here are some values of $\epsilon, \omega_\epsilon$ and N_ϵ.

ϵ	$\frac{3}{4}$	$\frac{7}{12}$	$\frac{6}{11}$	$\frac{1}{2}$	$\frac{34}{71}$	$\frac{5}{12}$	$\frac{1}{3}$
ω_ϵ	14	49	72	127	175	548	6460
N_ϵ	$e^{37.1101}$	$e^{204.75}$	$e^{335.71}$	$e^{679.585}$	$e^{1004.763}$	$e^{3894.57}$	e^{63727}

Thus $c < N^2$ which was conjectured in Granville and Tucker [GrTu02]. As a consequence of Theorem 1, we have

Theorem 2. *Assume Conjecture 1.2. Then the equation*

$$n(n+d)\cdots(n+(k-1)d) = by^\ell \tag{3}$$

in integers $n \ge 1, d > 1, k \ge 4, b \ge 1, y \ge 1, \ell > 1$ with $gcd(n, d) = 1$ and $P(b) \le k$ implies $\ell \le 7$. Further $k < e^{13006.2}$ when $\ell = 7$.

We observe that $e^{13006.2} < e^{e^{9.52}}$. Assuming $abc-$conjecture, Shorey [Sho99] proved that (3) with $\ell \ge 4$ implies that k is bounded by an absolute constant, the assertion for $\ell \in \{2, 3\}$ is due to Granville (see Laishram [Lai04, p. 69]). For a given $k \ge 3$, Győry, Hajdu and Saradha [GyHaSa04] showed that $abc-$conjecture implies that (3) has only finitely many solutions in positive integers $n, d > 1, b, y$ and $\ell \ge 4$. Saradha [Sar] showed that (3) with $k \ge 8$ implies that $\ell \le 29$ and further $k \le 8, 32, 10^2, 10^7$ and $e^{e^{280}}$ according as $\ell = 29$, $\ell \in \{23, 19\}, \ell = 17, 13$ and $\ell \in \{11, 7\}$, respectively. It has been conjectured that $(k, l) \in \{(3, 3), (4, 2), (3, 2)\}$ whenever there are positive integers $n, d > 1, y \ge 1, b, \ell \ge 2$ and $k \ge 3$ with $\gcd(n, d) = 1$ and $P(b) \le k$ satisfying (3) and it is known that (3) has infinitely many solutions when $(k, \ell) \in \{(3, 2), (3, 3)(4, 2)\}$. For an account of results on (3), we refer to Shorey [Sho02b], [Sho02a] and Shorey and Saradha [SaSh05].

Nagell-Ljunggren equation is the equation

$$y^q = \frac{x^n - 1}{x - 1} \tag{4}$$

in integers $x > 1, y > 1, n > 2, q > 1$. It is known that

$$11^2 = \frac{3^5 - 1}{3 - 1}, 20^2 = \frac{7^4 - 1}{7 - 1}, 7^3 = \frac{18^3 - 1}{18 - 1}$$

which are called the *exceptional solutions*. Any other solution is termed as *non-exceptional solutions*. For an account of results on (4), see Shorey [Sho99] and Bugeaud and Mignotte [BuMi02]. It is conjectured that there are no *non-exceptional solutions*. We prove in Section 7 the following.

Theorem 3. *Assume Conjecture 1.2. There are no non-exceptional solutions of equation* (4) *in integers* $x > 1, y > 1, n > 2, q > 1$.

Let $(p, q, r) \in \mathbb{Z}_{\geq 2}$ with $(p, q, r) \neq (2, 2, 2)$. The equation

$$x^p + y^q = z^r, \quad (x, y, z) = 1, x, y, z \in \mathbb{Z} \tag{5}$$

is called the *Generalized Fermat Equation* or *Fermat-Catalan Equation* with signature (p, q, r). An integer solution (x, y, z) is said to be non-trivial if $xyz \neq 0$ and primitive if x, y, z are coprime. We are interested in finding non-trivial primitive integer solutions of (5). The case $p = q = r$ is the famous *Fermat's equation* which is completely solved by Wiles [Wil95]. One of known solution $1^p + 2^3 = 3^2$ of (5) comes from *Catalan's equation*. Let $\chi = \frac{1}{p} + \frac{1}{q} + \frac{1}{r} - 1$. The parametrization of nontrivial primitive integer solutions for (p, q, r) with $\chi \geq 0$ is completely solved ([Beu04], [Coh07]). It was shown by Darmon and Granville [DaGr95] that (5) has only finitely many equations in x, y, z if $\chi < 0$.When $2 \in \{p, q, r\}$, there are some known solutions. So, we consider $p \geq 3, q \geq 3, r \geq 3$. An open problem in this direction is the following.

Conjecture 1.3. Tijdeman, Zagier: *There are no non-trivial solutions to* (5) *in positive integers* x, y, z, p, q, r *with* $p \geq 3, q \geq 3$ *and* $r \geq 3$.

This is also referred to as *Beal's Conjecture* or *Fermat-Catalan Conjecture*. This conjecture has been established for many signatures (p, q, r), including for several infinite families of signatures. For exhaustive surveys, see [Beu04], [Coh07, Chapter 14], [Kra99] and [PSS07]. Let $[p, q, r]$ denote all permutations of ordered triples (p, q, r) and let

$$Q = \{[3, 5, p] : 7 \leq p \leq 23, p \text{ prime}\} \cup \{[3, 4, p] : p \text{ prime}\}.$$

We prove the following in Section 8.

Theorem 4. *Assume Conjecture 1.2. There are no non-trivial solutions to* (5) *in positive integers* x, y, z, p, q, r *with* $p \geq 3, q \geq 3$ *and* $r \geq 3$ *with* $(p, q, r) \notin Q$. *Further for* $(p, q, r) \in Q$, *we have* $\max(x^p, y^q, z^r) < e^{1758.3353}$.

Another equation which we will be considering is the equation of Goormaghtigh

$$\frac{x^m - 1}{x - 1} = \frac{y^n - 1}{y - 1} \text{ integers } x > 1, y > 1, m > 2, n > 2 \text{ with } x \neq y. \tag{6}$$

We may assume without loss of generality that $x > y > 1$ and $2 < m < n$. It is known that

$$31 = \frac{5^3 - 1}{5 - 1} = \frac{2^5 - 1}{2 - 1} \text{ and } 8191 = \frac{90^3 - 1}{90 - 1} = \frac{2^{13} - 1}{2 - 1} \tag{7}$$

are the solutions of (6) and it is conjectured that there are no other solutions. A weaker conjecture states that there are only finitely many solutions x, y, m, n of (6). We refer to [Sho99] for a survey of results on (6). We prove in Section 9 that

Theorem 5. *Assume Conjecture 1.2. Then equation* (6) *in integers* $x > 1, y > 1, m > 2, n > 3$ *with* $x > y$ *implies that* $m \leq 6$ *and further* $7 \leq n \leq 17, n \notin \{11, 16\}$ *if* $m = 6$*; moreover there exists an effectively computable absolute constant* C *such that*

$$\max(x, y, n) \leq C.$$

Thus, assuming Conjecture 1.2, equation (6) has only finitely many solutions in integers $x > 1, y > 1, m > 2, n > 3$ with $x \neq y$ and this improves considerably Saradha [Sar, Theorem 1.4].

2. Notation and Preliminaries

For an integer $i > 0$, let p_i denote the $i-$th prime. For a real $x > 0$, let $\Theta(x) = \prod_{p \leq x} p$ and $\theta(x) = \log(\Theta(x))$. We write $\log_2 i$ for $\log(\log i)$. We have

Lemma 2.1. *We have*

(*i*) $\pi(x) \leq \dfrac{x}{\log x}\left(1 + \dfrac{1.2762}{\log x}\right)$ for $x > 1$.
(*ii*) $p_i \geq i(\log i + \log_2 i - 1)$ for $i \geq 1$
(*iii*) $\theta(p_i) \geq i(\log i + \log_2 i - 1.076869)$ for $i \geq 1$
(*iv*) $\theta(x) < 1.000081x$ for $x > 0$
(*v*) $\mathrm{ord}_p(k!) \geq \frac{k-p}{p-1} - \frac{\log(k-1)}{\log p}$ for $p < k$.
(*vi*) $\sqrt{2\pi k}(\frac{k}{e})^k e^{\frac{1}{12k+1}} \leq k! \leq \sqrt{2\pi k}(\frac{k}{e})^k e^{\frac{1}{12k}}$.

Here we understand that $\log_2 1 = -\infty$. The estimates (*i*) and (*ii*) are due to Dusart, see [Dus99b] and [Dus99a], respectively. The estimate (*iii*) is [Rob83, Theorem 6]. For estimate (*iv*), see [Dus99b]. For a proof of (*v*), see [LaSh04, Lemma 2(i)]. The estimate (*vi*) is [Rob55, Theorem 6].

3. Proof of Theorem 1

Let $\epsilon > 0$ and $N \geq 1$ be an integer with $\omega(N) = \omega$. Then $N \geq \Theta(p_\omega)$ or $\log N \geq \theta(p_\omega)$. Given ω, we observe that $\frac{M^\epsilon}{(\log M)^\omega}$ is an increasing function for $\log M \geq \frac{\omega}{\epsilon}$. Let

$$X_0(i) = \log i + \log_2 i - 1.076869.$$

Then $\theta(p_\omega) \geq \omega X_0(\omega)$ by Lemma 2.1 (*iii*). Observe that $X_0(i) > 1$ for $i \geq 5$. Let $\omega_1 \geq 5$ be smallest ω such that

$$\epsilon X_0(\omega) - \log X_0(\omega) \geq 1 \quad \text{for all } \omega \geq \omega_1. \tag{8}$$

Note that $\epsilon X_0(\omega) \geq 1$ for $\omega \geq \omega_1$ implying $\log N \geq \theta(p_\omega) \geq \omega X_0(\omega) \geq \frac{\omega}{\epsilon}$ for $\omega \geq \omega_1$ by Lemma 2.1 (*iii*). Therefore

$$\frac{\omega! N^\epsilon}{(\log N)^\omega} \geq \frac{\omega! \Theta(p_\omega)^\epsilon}{(\theta(p_\omega))^\omega} \geq \frac{\omega! e^{\epsilon\omega X_0(\omega)}}{(\omega X_0(\omega))^\omega} > \sqrt{2\pi\omega}(\frac{\omega}{e})^\omega \frac{e^{\epsilon\omega X_0(\omega)}}{(\omega X_0(\omega))^\omega} \text{ for } \omega \geq \omega_1.$$

Thus for $\omega \geq \omega_1$, we have from (8) that

$$\log\left(\frac{\omega! e^{\epsilon\omega X_0(\omega)}}{(\omega X_0(\omega))^\omega}\right) > \log\sqrt{2\pi\omega} + \omega(\log(\omega)-1) + \epsilon\omega X_0(\omega) - \omega(\log\omega + \log X_0(\omega))$$
$$> \log\sqrt{2\pi\omega} + \omega(\epsilon X_0(\omega) - \log X_0(\omega) - 1) \geq \log\sqrt{2\pi\omega}$$

implying

$$\frac{\omega! N^\epsilon}{(\log N)^\omega} \geq \frac{\omega!\Theta(p_\omega)^\epsilon}{(\theta(p_\omega))^\omega} > \sqrt{2\pi\omega} \text{ for } \omega \geq \omega_1.$$

Define ω_ϵ be the smallest $\omega \leq \omega_1$ such that

$$\theta(p_\omega) \geq \frac{\omega}{\epsilon} \text{ and } \frac{\omega!\Theta(p_\omega)^\epsilon}{(\theta(p_\omega))^\omega} > \sqrt{2\pi\omega} \text{ for all } \omega_\epsilon \leq \omega \leq \omega_1 \tag{9}$$

by taking the exact values of ω and θ. Then clearly

$$\frac{\omega! N^\epsilon}{(\log N)^\omega} \geq \frac{\omega!\Theta(p_\omega)^\epsilon}{(\theta(p_\omega))^\omega} > \sqrt{2\pi\omega} \text{ for } \omega \geq \omega_\epsilon. \tag{10}$$

Here are values of ω_ϵ for some ϵ values.

ϵ	$\frac{3}{4}$	$\frac{7}{12}$	$\frac{6}{11}$	$\frac{1}{2}$	$\frac{34}{71}$	$\frac{5}{12}$	$\frac{1}{3}$
ω_ϵ	14	49	72	127	175	548	6458

Let $\omega < \omega_\epsilon$ and $N \geq \Theta(\omega_\epsilon)$. Then $\log N \geq \theta(\omega_\epsilon) \geq \frac{\omega_\epsilon}{\epsilon}$. Therefore

$$\frac{\omega! N^\epsilon}{(\log N)^\omega} \geq \frac{\omega!\Theta(p_{\omega_\epsilon})^\epsilon}{(\theta(p_{\omega_\epsilon}))^\omega} = \frac{\omega_\epsilon!\Theta(p_{\omega_\epsilon})^\epsilon}{(\theta(p_{\omega_\epsilon}))^{\omega_\epsilon}} \cdot \frac{\omega!}{\omega_\epsilon!}(\theta(p_{\omega_\epsilon}))^{\omega_\epsilon-\omega} > \sqrt{2\pi\omega_\epsilon}\frac{\omega!\omega_\epsilon^{\omega_\epsilon-\omega}}{\omega_\epsilon!} \geq \sqrt{2\pi\omega_\epsilon}.$$

Combining this with (10), we obtain

$$\frac{(\log N)^\omega}{\omega!} < \frac{N^\epsilon}{\sqrt{2\pi\max(\omega,\omega_\epsilon)}} \leq \frac{N^\epsilon}{\sqrt{2\pi\omega_\epsilon}} \text{ for } N \geq \Theta(\omega_\epsilon). \tag{11}$$

Further we now prove

$$\frac{(\log N)^\omega}{\omega!} < \frac{5N^{\frac{3}{4}}}{6} \text{ for } N \geq 1. \tag{12}$$

For that we take $\epsilon = \frac{3}{4}$. Then $\omega_\epsilon = 14$ and we may assume that $N < \Theta(p_{14})$. Then $\omega = \omega(N) < 14$. Observe that $N \geq \Theta(p_\omega)$ and $\frac{N^{\frac{3}{4}}}{(\log N)^\omega}$ is increasing for $\log N \geq \frac{4\omega}{3}$. For $4 \leq \omega < 14$, we check that

$$\theta(p_\omega) \geq \frac{4\omega}{3} \text{ and } \frac{\omega!\Theta(p_\omega)^{\frac{3}{4}}}{(\theta(p_\omega))^\omega} > \frac{6}{5}$$

implying (12) when $4 \leq \omega = \omega(N) < 14$. Thus we may assume that $\omega = \omega(N) < 4$. We check that

$$\frac{\omega! N^{\frac{3}{4}}}{(\log N)^\omega} > \frac{6}{5} \text{ at } N = e^{\frac{4\omega}{3}} \tag{13}$$

for $1 \leq \omega < 4$ implying (12) for $N \geq e^{\frac{4\omega}{3}}$. Thus we may assume that $N < e^{\frac{4\omega}{3}}$. Then $N \in \{2,3\}$ if $\omega = \omega(N) = 1$, $N \in \{6,10,12,14\}$ if $\omega = \omega(N) = 2$ and $N \in \{30,42\}$ if

$\omega(N) = 3$. For these values of N too, we find that (13) is valid implying (12). Clearly (12) is valid when $N = 1$.

We now prove Theorem 1. Assume Conjecture 1.2. Let $\epsilon > 0$ be given. Let a, b, c be positive integers such that $a + b = c$ and $\gcd(a, b) = 1$. By Conjecture 1.2, $c \leq \frac{6}{5}N\frac{(\log N)^\omega}{\omega!}$ where $N = N(abc)$. Now assertion 2 follows from (12). Let $0 < \epsilon \leq \frac{3}{4}$ and $N_\epsilon = \Theta(p_{\omega_\epsilon})$. By (11), we have

$$c < \frac{6N^{1+\epsilon}}{5\sqrt{2\pi \max(\omega, \omega_\epsilon)}}.$$

The table is obtained by taking the table values of $\epsilon, \omega_\epsilon$ given after (10) and computing N_ϵ for those ϵ given in the table. Hence the Theorem. $\square$

4. Proof of Theorem 2

Let n, d, k, b, y be positive integers with $n \geq 1, d > 1, k \geq 4, b \geq 1, y \geq 1$, $\gcd(n, d) = 1$ and $P(b) \leq k$. We consider the Diophantine equation

$$n(n + d) \cdots (n + (k - 1)d) = by^\ell. \tag{14}$$

Observe that $P(n(n + d) \cdots (n + (k - 1)d)) > k$ by a result of Shorey and Tijdeman [ShTi90] and hence $P(y) > k$ and $n + (k - 1)d > (k + 1)^\ell$. For every $0 \leq i < k$, we write

$$n + id = A_i X_i^\ell \text{ with } P(A_i) \leq k \text{ and } (X_i, \prod_{p \leq k} p) = 1.$$

Without loss of generality, we may assume that $k = 4$ or $k \geq 5$ is a prime which we assume throughout in this section. We observe that $(A_i, d) = 1$ for $0 \leq i < k$ and $(X_i, X_j) = 1$. Let

$$S_0 = \{A_0, A_1, \ldots, A_{k-1}\}.$$

For every prime $p \leq k$ and $p \nmid d$, choose i_p be such that $\mathrm{ord}_p(A_i)$ $=\mathrm{ord}_p(n + id) \leq \mathrm{ord}_p(n + i_p d)$ for $0 \leq i < k$. For a $S \subset S_0$, let

$$S' = S - \{A_{i_p} : p \leq k, p \nmid d\}.$$

Then $|S'| \geq |S| - \pi_d(k)$. By Sylvester-Erdős inequality(see [ErSe75, Lemma 2] for example), we obtain

$$\prod_{A_i \in S'} A_i | (k - 1)! \prod_{p|d} p^{-\mathrm{ord}_p((k-1)!)}. \tag{15}$$

As a consequence, we have

Lemma 4.1. *Let $\alpha, \beta \in \mathbb{R}$ with $\alpha \geq 1$ and $e\beta < \alpha$. Let*

$$S_1 := S_1(\alpha) := \{A_i \in S_0 : A_i \leq \alpha k\}.$$

For

$$k \geq \frac{\log(\frac{e\alpha}{\sqrt{\beta}}) + \frac{k \log(\alpha k)}{\log k}\left(1 + \frac{1.2762}{\log k}\right) - \log(\alpha k)}{\log(e\alpha) + \beta \log\left(\frac{\beta}{e\alpha}\right)}, \tag{16}$$

we have $|S_1| > \beta k$.

Proof. Let $S = S_0$, $s_1 = |S_1|$ and $s_2 = |S' - S_1|$. Then $s_2 \geq k - \pi(k) - s_1$. We get from (15) that

$$s_1! \prod_{i=1}^{k-\pi(k)-s_1} ([\alpha k + i]) \leq \prod_{A_i \in S'} A_i \leq (k-1)! \tag{17}$$

since elements of $S' - S_1$ are distinct. Using Lemma 2.1 (vi), we obtain

$$(\alpha k)^{k-\pi(k)} < \frac{(k-1)!}{s_1!}(\alpha k)^{s_1} < \begin{cases} \sqrt{2\pi(k-1)}\left(\frac{k-1}{e}\right)^{k-1} e^{\frac{1}{12(k-1)}} & \text{if } s_1 = 0 \\ \sqrt{\frac{k-1}{s_1}}\left(\frac{\alpha k e}{s_1}\right)^{s_1}\left(\frac{k-1}{e}\right)^{k-1} & \text{if } s_1 > 0. \end{cases}$$

We check that the expression for $s_1 = 0$ is less than that of $s_1 = 1$ since $\alpha \geq 1$. Suppose $s_1 \leq \beta k$. Observe that

$$\sqrt{\frac{k-1}{s_1}}(\frac{\alpha k e}{s_1})^{s_1}$$

is an increasing function of s_1 since $s_1 \leq \beta k$ and $e\beta < \alpha$. This can be verified by taking log of the above expression and differentiating it with respect to s_1. Therefore

$$(\alpha k)^{k-\pi(k)} < \sqrt{\frac{k-1}{\beta k}}\left(\frac{e\alpha}{\beta}\right)^{\beta k}\left(\frac{k-1}{e}\right)^{k-1} < \sqrt{\frac{1}{\beta}}\left(\frac{e\alpha}{\beta}\right)^{\beta k}\left(\frac{k}{e}\right)^{k-1}$$

implying

$$(e\alpha)^k\left(\frac{\beta}{e\alpha}\right)^{\beta k} < \frac{e\alpha}{\sqrt{\beta}}(\alpha k)^{\pi(k)-1}.$$

Using Lemma 2.1 (i), we obtain

$$\log(e\alpha) + \beta\log\left(\frac{\beta}{e\alpha}\right) < \frac{1}{k}\log(\frac{e\alpha}{\sqrt{\beta}}) + \frac{\log(\alpha k)}{\log k}\left(1 + \frac{1.2762}{\log k}\right) - \frac{\log(\alpha k)}{k}.$$

The right hand side of the above inequality is a decreasing function of k for k given by (16). This can be verified by observing that $\frac{\log \alpha k}{\log k} = 1 + \frac{\log \alpha}{\log k}$ and differentiating $\frac{1.2762+\log \alpha}{\log k} - \frac{\log(\alpha k)}{k}$ with respect to k. This is a contradiction for k given by (16). □

Corollary 4.2. *For* $k > 113$*, there exist* $0 \leq f < g < h < k$ *with* $h - f \leq 8$ *such that* $max(A_f, A_g, A_h) \leq 4k$.

Proof. By dividing $[0, k-1]$ into subintervals of the form $[9i, 9(i+1))$, it suffices to show $S_1(4) > 2([\frac{k}{9}] + 1)$ where S_1 is as defined in Lemma 4.1. Taking $\alpha = 4, \beta = \frac{1}{4}$, we obtain from Lemma 4.1 that for $k \geq 700$, $|S_1(4)| > \frac{k}{4} > 2([\frac{k}{9}] + 1)$. Thus we may suppose $k < 700$ and $|S_1(4)| \leq 2([\frac{k}{9}] + 1)$. For each prime k with $113 < k < 700$, taking $\alpha = 4$ and $\beta k = 2([\frac{k}{9}] + 1)$ in Lemma 4.1, we get a contradiction from (17). Therefore $|S_1(4)| > 2([\frac{k}{9}] + 1)$ and the assertion follows. □

Given $0 \le f < g < h \le k-1$, we have

$$(h-f)A_gX_g^\ell = (h-g)A_fX_f^\ell + (g-f)A_hX_h^\ell. \tag{18}$$

Let $\lambda =$gcd$(h-f, h-g, g-f)$ and write $h-f = \lambda w, h-g = \lambda u, g-f = \lambda v$. Rewriting $h-f = h-g+g-f$ as

$$w = u + v \text{ with } \gcd(u,v) = 1,$$

(18) can be written as

$$wA_gX_g^\ell = uA_fX_f^\ell + vA_hX_h^\ell. \tag{19}$$

Let $G = \gcd(wA_g, uA_f, vA_h)$,

$$r = \frac{uA_f}{G}, s = \frac{vA_h}{G}, t = \frac{wA_g}{G} \tag{20}$$

and we rewrite (19) as

$$tX_g^\ell = rX_f^\ell + sX_h^\ell. \tag{21}$$

Note that $\gcd(rX_f^\ell, sX_h^\ell) = 1$.

From now on, we assume explicit $abc-$conjecture. Given $\epsilon > 0$, let $N(rstX_fX_gX_h) \ge N_\epsilon$ which we assume from now on till the expression (27). By Theorem 1, we obtain

$$tX_g^\ell < \kappa_\epsilon N(rstX_fX_gX_h)^{1+\epsilon} \tag{22}$$

i.e.,

$$X_g^\ell < \kappa_\epsilon \frac{N(rst)^{1+\epsilon}(X_fX_gX_h)^{1+\epsilon}}{t}. \tag{23}$$

Here $N_\epsilon = \kappa_\epsilon = 1$ if $\epsilon \ge \frac{3}{4}$ and we may also take $\kappa_{\frac{3}{4}} \le \frac{6}{5\sqrt{28\pi}}$ if $N(rstX_fX_gX_h) \ge N_{\frac{3}{4}}$. We will be taking $\epsilon = \frac{3}{4}$ for $\ell > 7$ and $\epsilon \in \{\frac{5}{12}, \frac{1}{3}\}$ for $\ell = 7$. We have from (22) that

$$rst(X_fX_gX_h)^\ell < \kappa_\epsilon^3 N(rst)^{3(1+\epsilon)}(X_fX_gX_h)^{3(1+\epsilon)}.$$

Putting $X^3 = X_fX_gX_h$, we obtain

$$X^{\ell-3(1+\epsilon)} < \kappa_\epsilon N(rst)^{\frac{2}{3}+\epsilon} = \kappa_\epsilon N(\frac{uvwA_fA_gA_h}{G^3})^{\frac{2}{3}+\epsilon}. \tag{24}$$

Again from (21), we have

$$rs(X_fX_h)^\ell \le \left(\frac{rX_f^\ell + sX_h^\ell}{2}\right)^2 = \frac{t^2X_g^{2\ell}}{4}$$

implying

$$X_fX_hX_g \le \left(\frac{t^2}{4rs}\right)^{\frac{1}{\ell}} X_g^3 = \left(\frac{w^2A_g^2}{4uvA_fA_h}\right)^{\frac{1}{\ell}} X_g^3.$$

Therefore we have from (23) that

$$X_g^\ell < \kappa_\epsilon \frac{N(rst)^{1+\epsilon}X_g^{3+3\epsilon}}{t}\left(\frac{t^2}{4rs}\right)^{\frac{1+\epsilon}{\ell}} = \kappa_\epsilon \frac{N(rst)^{1+\epsilon}X_g^{3+3\epsilon}}{(4rst)^{\frac{1+\epsilon}{\ell}}t^{1-\frac{3(1+\epsilon)}{\ell}}} \tag{25}$$

i.e.,

$$X_g^{\ell-3(1+\epsilon)} < \kappa_\epsilon \frac{N(rst)^{(1+\epsilon)(1-\frac{1}{\ell})}}{4^{\frac{1+\epsilon}{\ell}} t^{1-\frac{3(1+\epsilon)}{\ell}}} = \kappa_\epsilon \frac{N(\frac{uvwA_fA_gA_h}{G^3})^{(1+\epsilon)(1-\frac{1}{\ell})}}{4^{\frac{1+\epsilon}{\ell}}(\frac{wA_g}{G})^{1-\frac{3(1+\epsilon)}{\ell}}}. \tag{26}$$

Observe that

$$\frac{N(rst)^{(1+\epsilon)(1-\frac{1}{\ell})}}{4^{\frac{1+\epsilon}{\ell}} t^{1-\frac{3(1+\epsilon)}{\ell}}} \le \frac{N(rs)^{(1+\epsilon)(1-\frac{1}{\ell})} N(t)^{\epsilon+\frac{2(1+\epsilon)}{\ell}}}{4^{\frac{1+\epsilon}{\ell}}}.$$

Hence we also have from (26) that

$$X_g^{\ell-3(1+\epsilon)} < \kappa_\epsilon \frac{N(\frac{uvA_fA_h}{G^2})^{(1+\epsilon)(1-\frac{1}{\ell})} N(\frac{wA_g}{G})^{\epsilon+\frac{2(1+\epsilon)}{\ell}}}{4^{\frac{1+\epsilon}{\ell}}}. \tag{27}$$

Lemma 4.3. *Let $\ell \ge 11$. Let $S_0 = \{A_0, A_1, \ldots, A_{k-1}\} = \{B_0, B_1, \ldots, B_{k-1}\}$ with $B_0 \le B_1 \le \ldots \le B_{k-1}$. Then*

$$B_0 \le B_1 < B_2 \ldots < B_{k-1}.$$

In particular $|S_0| \ge k-1$.

Proof. Suppose there exists $0 \le f < g < h < k$ with $\{f, g, h\} = \{i_1, i_2, i_3\}$ and

$$A_{i_1} = A_{i_2} = A \text{ and } A_{i_3} \le A.$$

By (19) and (20), we see that $\max(A_f, A_g, A_h) \le G$ and therefore $r \le u < k, s \le v < k$ and $t \le w < k$. Since $X_g > k$, we get from the first inequality of (26) with $\epsilon = \frac{3}{4}, N_\epsilon = \kappa_\epsilon = 1$ that

$$k^{\ell-3(1+\epsilon)} < (rs)^{(1+\epsilon)(1-\frac{1}{\ell})} t^{\epsilon+\frac{2(1+\epsilon)}{\ell}} < k^{2+3\epsilon}$$

implying $\ell < 5 + 6\epsilon = 5 + \frac{9}{2}$. This is a contradiction since $\ell \ge 11$. Therefore either A_i's are distinct or if $A_i = A_j = A$, then $A_m > A$ for $m \notin \{i, j\}$ implying the assertion. □

As a consequence, we have

Corollary 4.4. *Let d be even and $\ell \ge 11$. Then $k \le 13$.*

Proof. Let d be even and $\ell \ge 11$. Then we get from (15) with $S = S_0$ that

$$\prod_{A_i \in S'} A_i \le (k-1)! 2^{\mathrm{ord}_2((k-1)!)} = \prod_{2i+1 \le k-1} (2i+1).$$

On the other hand, since $\gcd(n, d) = 1$, we see that all A_i's are odd and $|S'| \ge |S_0| - \pi(k) \ge k - 1 - \pi(k)$ by Lemma 4.3. Hence

$$\prod_{A_i \in S'} A_i \ge \prod_{i=1}^{k-1-\pi(k)} (2i-1).$$

This is a contradiction since $2(k-1-\pi(k)) > k-1$ for $k \ge 14$. □

Lemma 4.5. *Let $\ell \ge 11$. Then $k < 400$.*

Proof. Assume that $k \geq 400$. By Corollary 4.4, we may suppose that d is odd. Further by Corollary 4.2, there exists $f < g < h$ with $h - f \leq 8$ and $\max(A_f, A_g, A_h) \leq 4k$. Since $n + (k-1)d > k^\ell$, we observe that $X_f > k, X_g > k, X_h > k$ implying $X > k$. First assume that $N = N(rstX_fX_gX_h) < e^{37.12}$. Then taking $\epsilon = \frac{3}{4}, N_\epsilon = 1$ in (22), we get $400^{11} \leq k^{11} \leq tX_g^\ell < N^{1+\frac{3}{4}} \leq e^{37.12(1+\frac{3}{4})}$ which is a contradiction. Hence we may suppose that $N \geq e^{37.12} \geq N_{\frac{3}{4}}$.

Note that we have $u + v = w \leq h - f \leq 8$. We observe that uvw is even. If $A_fA_gA_h$ is odd, then $h - f, g - f, h - g$ are all even implying $1 \leq u, v, w \leq 4$ or $N(uvw) \leq 6$ giving $N(uvwA_fA_gA_h) \leq 6A_fA_gA_h$. Again if $A_fA_gA_h$ is even, then $N(uvwA_fA_gA_h) \leq N((uvw)')A_fA_gA_h \leq 35A_fA_gA_h$ where $(uvw)'$ is the odd part of uvw and $N((uvw)') \leq 35$. Observe that $N((uvw)')$ is obtained when $w = 7, u = 2, v = 5$ or $w = 7, u = 5, v = 2$. Thus we always have $N(uvwA_fA_gA_h) \leq 35A_fA_gA_h \leq 35 \cdot (4k)^3$ since $\max(A_f, A_g, A_h) \leq 4k$. Therefore taking $\epsilon = \frac{3}{4}$ in (24), we obtain using $\ell \geq 11$ and $X > k$ that

$$k^{11-3(1+\frac{3}{4})} < \frac{6}{5\sqrt{28\pi}} 35^{\frac{2}{3}+\frac{3}{4}} (4k)^{3(\frac{2}{3}+\frac{3}{4})}.$$

This is a contradiction since $k \geq 400$. Hence the assertion. □

5. Proof of Theorem 2 for $4 \leq k < 400$

We assume that $\ell \geq 11$. It follows from the result of Saradha and Shorey [SaSh05, Theorem 1] that $d > 10^{15}$. Hence we may suppose that $d > 10^{15}$ in this section.

Lemma 5.1. *Let* $r_k = [k + 1 - \pi(k) - \frac{\sum_{i \leq k} \log i}{15 \log 10}]$ *and*

$$I(k) = \{i \in [1, k] : P(n + id) > k\}.$$

Then $|I(k)| \geq r_k$.

Proof. Suppose not. Then $|I(k)| \leq r_k - 1$. Let

$$I'(k) = \{i \in [1, k] : P(n + id) \leq k\} = \{i \in [1, k] : n + id = A_i\}.$$

We have $A_i = n + id \geq (n + d)$ for $i \in I'(k)$. Let $S = \{A_i : i \in I'(k)\}$. Then $|S| \geq k + 1 - r_k$. From (15), we get

$$(k-1)! \geq \prod_{A_i \in S'} A_i \geq (n + d)^{|S'|} > d^{k+1-r_k-\pi(k)}.$$

Since $d > 10^{15}$, we get

$$k + 1 - \pi(k) - \frac{\sum_{i \leq k} \log i}{15 \log 10} < r_k = [k + 1 - \pi(k) - \frac{\sum_{i \leq k} \log i}{15 \log 10}].$$

This is a contradiction. □

Here are some values of (k, r_k).

k	7	11	13	17	18	28	30	36
r_k	3	6	7	10	10	18	18	23

We give the strategy here. Let $I_k = [0, k-1] \cap \mathbb{Z}$ and a_0, b_0, z_0 be given. Let obtain a subset $I_0 \subseteq I_k$ with the following properties:

(1) $|I_0| \geq z_0 \geq 3$.
(2) $P(A_i) \leq a_0$ for $i \in I_0$.
(3) $I_0 \subseteq [j_0, j_0 + b_0 - 1]$ for some j_0.
(4) $X_0 = \max_{i \in I_0}\{X_i\} > k$ and let $i_0 \in I_0$ be such that $X_0 = X_{i_0}$.

For any $i, j \in I_0$, taking $\{f, g, h\} = \{i, j, i_0\}$, let $N = N(rstX_fX_gX_h)$. Observe that $X_0 \geq p_{\pi(k)+1}$ and further for any $f, g, h \in I_0$, we have $N(uvw) \leq \prod_{p \leq b_0 - 1} p$ and $N(A_fA_gA_h) \leq \prod_{p \leq a_0} p$. We will always take $\epsilon = \frac{3}{4}, N_\epsilon = 1$ so that $\kappa_\epsilon = 1$ in (22) to (27).

Case I: Suppose there exists $i, j \in I_0$ such that $X_i = X_j = 1$. Taking $\{f, g, h\} = \{i, j, i_0\}$ and $\epsilon = \frac{3}{4}$, we obtain from (23) and $\ell \geq 11$ that

$$p_{\pi(k)+1}^{\frac{37}{7}} \leq X_0^{\frac{\ell}{1+\frac{3}{4}} - 1} < N(uvwA_fA_gA_h) \leq \prod_{p \leq \max\{a_0, b_0 - 1\}} p. \tag{28}$$

Case II: There is at most one $i \in I_0$ such that $X_i = 1$. Then $|\{i \in I_0 : X_i > k\}| \geq z_0 - 1$. We take a_1, b_1, z_1 and find a subset $U_0 \subset I_0$ with the following properties:

(1) $|U_0| \geq z_1 \geq 3$, $\frac{z_0}{2} \leq z_1 \leq z_0$.
(2) $P(A_i) \leq a_1$ for $i \in U_0$.
(3) $U_0 \subseteq [i, i + b_1 - 1]$ for some i.

Let $X_1 = \max_{i \in U_0}\{X_i\} \geq p_{\pi(k)+z_1-1}$ and i_1 be such that $X_{i_1} = X_1$. Taking $\{f, g, h\} = \{i, j, i_1\}$ for some $i, j \in U_0$ and $\epsilon = \frac{3}{4}$, we obtain from (26) and $\ell \geq 11$ that

$$p_{\pi(k)+z_1-1}^{\frac{23}{7}} \leq X_0^{\frac{\ell}{1+\frac{3}{4}} - 3} < N(uvwA_fA_gA_h) \leq \prod_{p \leq \max\{a_1, b_1 - 1\}} p \tag{29}$$

since $\ell \geq 11$. One choice is $(U_0, a_1, b_1, z_1) = (I_0, a_0, b_0, z_0)$. We state the other choice.

Let $b' = \max(a_0, b_0 - 1)$. For each $\frac{b_0}{2} - 1 < p \leq b' - 1$, we remove those $i \in I_0$ such that $p|(n + id)$. There are at most $2(\pi(b' - 1) - \pi(\frac{b_0}{2} - 1))$ such i. Let I'_0 be obtained from I_0 after deleting those i's. Then $|I'_0| \geq z_0 - 2(\pi(b' - 1) - \pi(\frac{b_0}{2} - 1))$. Let

$$U_1 = I'_0 \cap [j_0, j_0 + \frac{b_0}{2} - 1] \text{ and } U_1 = I'_0 \cap [j_0 + \frac{b_0}{2}, j_0 + b_0 - 1].$$

Let $U_0 \in \{U_1, U_2\}$ for which $|U_i| = \max(|U_1|, |U_2|)$ and choose one of them if $|U_1| = |U_2|$. Then $|U_0| \geq \lceil \frac{z_0}{2} \rceil - \pi(b' - 1) + \pi(\frac{b_0}{2} - 1) = z_1$. Further $P(A_i) \leq \frac{b_0}{2} - 1 = a_1$ and $b_1 = \frac{b_0}{2}$. Further $X_1 = \max_{i \in U_0}\{X_i\} \geq p_{\pi(k)+z_1-1}$. Our choice of z_0, a_0, b_0 will imply that $z_1 \geq 3$.

4.1. $k \in \{4, 5, 7, 11\}$

We take $I_0 = U_0 = I_k, a_i = b_i = z_i = k$ for $i \in \{0, 1\}$ and hence $N(uvwA_fA_gA_h) \le \prod_{p\le k} p$. And the assertion follows since both (28) and (29) are contradicted.

4.2. $k \in \{13, 17, 19, 23\}$

We take $I_0 = \{i \in [1, 11] : p \nmid (n+id)$ for $13 \le p \le 23\}$. Then by $r_{11} = 6$ and Lemma 5.1 with $k = 11$, we see that $|I_0| \ge z_0 = 11 - 4 > 11 - r_{11} \ge 11 - |I(11)|$. Therefore there exist an $i \in I_0 \cap I_{11}$ and hence $X_i > 23$. We take $U_0 = I_0$, $a_i = b_i = 11, z_1 = z_0$ for $i \in \{0, 1\}$ and hence $N(uvwA_fA_gA_h) \le \prod_{p\le 11} p$. And the assertion follows since both (28) and (29) are contradicted.

4.3. $29 \le k \le 47$

We take $I_0 = \{i \in [1, 17] : p \nmid (n + id)$ for $17 \le p \le k\}$. Then by $r_{17} = 10$ and Lemma 5.1 with $k = 17$, we have $|I_0| \ge z_0 = 17 - (\pi(k) - \pi(13)) = 23 - \pi(k) \ge 23 - \pi(47) = 8 > 17 - r_{17} \ge 17 - |I(17)|$ implying that there exists $i \in I_0$ with $X_i > k$. We take $a_i = 13, b_i = 17, z_i = 23 - \pi(k)$ for $i \in \{0, 1\}$ and hence $N(uvwA_fA_gA_h) \le \prod_{p\le 13} p$. And the assertion follows since both (28) and (29) are contradicted.

4.4. $k \ge 53$

Given m and q such that $mq < k$, we consider the q intervals

$$I_j = [(j-1)m + 1, jm] \cap \mathbb{Z} \text{ for } 1 \le j \le q$$

and let $I' = \cup_{j=1}^{q} I_j$ and $I'' = \{i \in I' : m \le P(A_i) \le k\}$. There is at most one $i \in I'$ such that $mq - 1 < P(A_i) \le k$ and for each $2 \le j \le q$, there are at most j number of $i \in I'$ such that $\frac{mq-1}{j} < P(A_i) \le \frac{mq-1}{j-1}$. Therefore

$$|I''| \le \pi(k) - \pi(mq-1) + \sum_{j=2}^{q} j\left(\pi(\frac{mq-1}{j-1}) - \pi(\frac{mq-1}{j})\right)$$

$$= \pi(k) + \sum_{j=1}^{q-1} \pi(\frac{mq-1}{j}) - q\pi(m-1) =: T(k, m, q).$$

Hence there is at least one j such that $|I_j \cap I''| \le [\frac{T(k,m,q)}{q}]$. We will choose q such that $[\frac{T(k,m,q)}{q}] < r_m$. Let $I_0 = I_j \setminus I''$ and let j_0 be such that $I_0 \subseteq [(j_0-1)m+1, j_0m]$. Then $p|(n + id)$ imply $p < m$ or $p > k$ whenever $i \in I_0$. Further $|I_0| \ge z_0 = m - [\frac{T(k,m,q}{q}]$. Since $[\frac{T(k,m,q)}{q}] < r_m$, we get from Lemma 5.1 with $k = m$ and $n = (j_0 - 1)m$ that there is an $i \in I_0$ with $X_i > k$. Further $P(A_i) < m$ for all $i \in I_0$. Here are the choices of m and q.

k	$53 \le k < 89$	$89 \le k < 179$	$179 \le k < 239$	$239 \le k < 367$	$367 \le k < 433$
(m,q)	$(17,3)$	$(28,3)$	$(36,5)$	$(36,6)$	$(36,10)$

We have $a_0 = m-1, b_0 = m$ and $z_0 = m - [\frac{T(k,m,q)}{q}]$ and we check that $z_0 \ge 3$. The Subsection 4.3($29 \le k \le 47$) is in fact obtained by considering $m = 17, q = 1$. Now we consider Cases I and II and try to get contradiction in both (28) and (29). For these choices of (m,q), we find that the Cases I are contradicted. Further taking $U_0 = I_0, a_1 = a_0 = m-1, b_1 = b_0 = m, z_1 = z_0$, we find that Case II is also contradicted for $53 \le k < 89$. Thus the assertion follows in the case $53 \le k < 89$. So, we consider $k \ge 89$ and try to contradict Cases II. Recall that we have $X_i > k$ for all but at most one $i \in I_0$. Write $I_0 = U_1 \cup U_2$ where $U_1 = I_0 \cap [(j_0-1)m+1, (j_0-1)m+\frac{m}{2}]$ and $U_2 = I_0 \cap [(j_0-1)m+\frac{m}{2}+1, j_0 m]$. Let $U_0' = U_1$ or $U_0' = U_2$ according as $|U_1| \ge \frac{z_0}{2}$ or $|U_2| \ge \frac{z_0}{2}$, respectively. Let $U_0 = \{i \in U_0' : p \nmid A_i \text{ for } \frac{m}{2} \le p < m\}$. Then $|U_0| \ge z_1 := \frac{z_0}{2} - (\pi(m-1) - \pi(\frac{m}{2})) = \frac{m-[\frac{T(k,m,q)}{q}]}{2} - (\pi(m-1) - \pi(\frac{m}{2})) \ge 3$. Further $p|(n+id)$ with $i \in U_0$ imply $p < \frac{m}{2}$ or $p > k$. Now we have Case II with $a_1 = \frac{m}{2} - 1, b_1 = \frac{m}{2}$ and find that (29) is contradicted. Hence the assertion.

6. $\ell = 7$

Let $\ell = 7$. Assume that $k \ge exp(13006.2)$. Taking $\alpha = 3, \beta = \frac{1}{15} + \frac{2}{9}$ in Lemma 4.1, we get

$$|S_1(3)| = \{i \in [0, k-1] : A_i \le 3k\}| > k(\frac{1}{15} + \frac{2}{9}).$$

For i's such that $A_i \in S_1(3)$, we have $X_i > k$ and we arrange these X_i's in increasing order as $X_{i_1} < X_{i_2} < \ldots <$. Then $X_{i_j} \ge p_{\pi(k)+j}$. Consider the set $J_0 = \{i : X_i \ge p_{\pi(k)+[\frac{k}{15}]-2}\}$. We have

$$|J_0| > k(\frac{1}{15} + \frac{2}{9}) - \frac{k}{15} + 2 \ge 2\left([\frac{k-1}{9}] + 1\right).$$

Hence there are $f, g, h \in J_0$, $f < g < h$ such that $h - f \le 8$. Also $A_i \le 3k$ and $X = (X_f X_g X_h)^{\frac{1}{3}} \ge p_{\pi(k)+[\frac{k}{15}]-2}$.

First assume that $N = N(rstX_fX_gX_g) \ge exp(63727) \ge N_{\frac{1}{3}}$. Observe that $uvw \le 70$ since $2 \le u + v = w \le 8$, obtained at $2 + 5 = 7$. Taking $\epsilon = \frac{1}{3}$, we obtain from (24) and $\max(A_f, A_g, A_h) \le 3k$ that

$$p^3_{\pi(k)+[\frac{k}{15}]-2} < \frac{5}{6\sqrt{2\pi \cdot 6458}} N(uvwA_fA_gA_h) \le \frac{5 \cdot 70 \cdot (3k)^3}{6\sqrt{12920\pi}}.$$

Since $\pi(k) > 2$ we have $\pi(k) + [\frac{k}{15}] - 2 > \frac{k}{15}$ and hence $p_{\pi(k)+[\frac{k}{15}]-2} > \frac{k}{15}\log\frac{k}{15}$ by Lemma 2.1 (ii). Therefore

$$\left(\log\frac{k}{15}\right)^3 < \frac{350 \cdot (3 \cdot 15)^3}{6\sqrt{12920\pi}} \text{ or } k < 15 \cdot exp\left(45 \cdot \left(\frac{350}{6\sqrt{12920\pi}}\right)^{\frac{1}{3}}\right)$$

which is a contradiction since $k \geq exp(13006.2)$.

Therefore we have $N = N(rstX_fX_gX_h) < exp(63727)$. We may also assume that $N > exp(3895)$ otherwise taking $\epsilon = \frac{3}{4}$ in (22), we get $k^7 < X_g^7 < N^{1+\frac{3}{4}} \leq exp(3895 \cdot \frac{7}{4})$ or $k < exp(\frac{3895}{4})$ which is a contradiction. Now we take $\epsilon = \frac{5}{12}$ in (22) to get $k^7 < X_g^7 < N^{1+\frac{5}{12}} \leq exp(64266 \cdot \frac{17}{12})$ or $k < exp(13006.2)$. Hence the assertion.

7. Nagell-Ljungrenn equation: Proof of Theorem 3

Let $x > 1, y > 1, n > 2$ and $q > 1$ be a non-exceptional solution of (4). It was proved by Ljunggren [Lju43] that there are no further solutions of (4) when $q = 2$. Thus we may suppose that $q \geq 3$. Further it has been proved that $4 \nmid n$ by Nagell [Nag20], $3 \nmid n$ by Ljunggren [Lju43] and $5 \nmid n, 7 \nmid n$ by Bugeaud, Hanrot and Mignotte [BHM02]. Therefore $n \geq 11$. From (4), we get

$$1 + (x-1)y^q = x^n.$$

Then $y < x^{\frac{n}{q}} \leq x^{\frac{n}{3}}$ since $q \geq 3$ implying $N = N(x(x-1)y) < x^2y < x^{2+\frac{n}{3}}$. From (2) in Theorem 1, we obtain

$$x^n < N^{\frac{7}{4}} < x^{\frac{7}{2}+\frac{7n}{12}} \text{ implying } n < \frac{7}{2} + \frac{7n}{12}.$$

This gives $n \leq 8$ which is a contradiction.

8. Fermat-Catalan Equation

We may assume that each of p, q, r is either 4 or an odd prime. Let $[p, q, r]$ denote all permutations of ordered triple (p, q, r). The Fermat's Last Theorem (p, p, p) was proved by Wiles [Wil95]; $[3, p, p], [4, p, p]$ for $p \geq 7$ by Darmon and Merel [DaGr95] and $[3, 5, 5], [4, 5, 5]$ by Poonen; $[4, 4, p]$ by Bennett, Ellenberg, Ng [BEN10]. The signatures $[3, 3, p]$ for $p \leq 10^9$ was solved by Chen and Siksek [ChSi09], $[3, 4, 5]$ by Siksek and Stoll [SiSt90] and $[3, 4, 7]$ by Poonen, Schefer and Stoll [PSS07]. Hence we may suppose (p, q, r) is different from those values.

We may assume that $x > 1, y > 1, z > 1$. Then

$$x < z^{\frac{r}{p}}, y < z^{\frac{r}{q}}.$$

Given $\epsilon > 0$, by Theorem 1, we have

$$z^r < \begin{cases} N_\epsilon^{\frac{7}{4}} & \text{if } N(xyz) < N_\epsilon \\ N(xyz)^{1+\epsilon} \leq (xyz)^{1+\epsilon} & \text{if } N(xyz) \geq N_\epsilon. \end{cases} \tag{30}$$

In particular, taking $\epsilon = \frac{3}{4}$, we get

$$z^r < (xyz)^{\frac{7}{4}} < z^{\frac{7}{4}(1+\frac{r}{p}+\frac{r}{q})}$$

implying

$$\frac{4}{7} < \frac{1}{p} + \frac{1}{q} + \frac{1}{r}. \tag{31}$$

Thus we need to consider $[3, 3, p]$ for $p > 10^9$ and $(p, q, r) \in Q$. Let $\epsilon = \frac{34}{71}$. First assume that $N(xyz) \geq N_\epsilon$. Then

$$z^r < (xyz)^{1+\epsilon} < z^{(1+\epsilon)(1+\frac{r}{p}+\frac{r}{q})}$$

implying

$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} > \frac{1}{1+\epsilon} = \frac{71}{105} = \frac{1}{3} + \frac{1}{5} + \frac{1}{7}.$$

Therefore we may suppose that $N(xyz) < N_{\frac{34}{71}}$. Then from (30) that $\max(x^p, y^q, z^r) < N_{\frac{34}{71}}^{\frac{7}{4}} \leq e^{1758.3353}$ implying x, y, z, p, q, r are all bounded. This will imply that $[3, 3, p]$ with $p > 10^9$ does not have any solution. Hence the assertion. □

9. Goormaghtigh Equation

Let $d = \gcd(x, y)$. From (6), we have

$$x^{m-1} + \cdots + x = y^{n-1} + \cdots + y$$

implying $\mathrm{ord}_p(x) = \mathrm{ord}_p(y)$ for all primes $p|d$. Further

$$\sum_{i=1}^{m-1}(x^i - y^i) = (x-y)\left\{1 + \sum_{i=2}^{m-1}\frac{x^i - y^i}{x-y}\right\} = y^{n-1} + \cdots + y^m$$

which is

$$1 + \sum_{i=2}^{m-1}\frac{x^i - y^i}{x-y} = \frac{y^m}{x-y}\frac{y^{n-m}-1}{y-1}.$$

We observe that d is coprime to $\frac{y^{n-m}-1}{y-1}$ and also to the left hand side. Therefore

$$\mathrm{ord}_p(x-y) = m \cdot \mathrm{ord}_p(x) = m \cdot \mathrm{ord}_p(y) = m \cdot \mathrm{ord}_p(d)$$

for every prime $p|d$. Let $d_2 = \gcd(y-1, x-1, x-y)$ and d_3 be given by $x - y = d^m d_2 d_3$. We observe that $d_2 d_3 = 1$ if $n = m+1$ and $d_2 d_3 | (y+1)$ if $n = m+2$. We now rewrite (6) as

$$\frac{(y-1)x^m}{d^m d_2} + d_3 = \frac{(x-1)y^n}{d^m d_2}. \tag{32}$$

Let

$$N = N\left(\frac{x^m y^n (x-1)(y-1)d_3}{d^{2m} d_2^2}\right) \leq N(xy(x-1)(y-1)d_3) \leq \frac{xy(x-1)(y-1)d_3}{2^\delta d d_2}$$

where $\delta = 0$ if $2|dd_2$ and 1 otherwise. Recall that $d = \gcd(x, y)$ and $d_2|(x-1)$. Let $\epsilon < \frac{3}{4}$. We obtain from (32) and Theorem 1 and $x - y = d^m d_2 d_3$ that

$$\max\{\frac{(y-1)x^m d_3}{(x-y)}, \frac{(x-1)y^n d_3}{x-y}\} < \begin{cases} N_\epsilon^{\frac{7}{4}} & \text{if } N < N_\epsilon \\ N^{1+\epsilon} & \text{if } N \geq N_\epsilon. \end{cases} \tag{33}$$

Assume that $N \geq N_\epsilon$. Then we obtain using (33) that

$$x^m < x^{2+2\epsilon} y^{1+2\epsilon}(x-y)\frac{d_3^\epsilon}{(2^\delta dd_2)^{1+\epsilon}} < x^{4+5\epsilon} \tag{34}$$

$$y^n < x^{1+2\epsilon} y^{1+\epsilon}(y-1)^{1+\epsilon}(x-y)\frac{d_3^\epsilon}{(2^\delta dd_2)^{1+\epsilon}}. \tag{35}$$

since $y < x$ and $d_3 \leq x - y < x$. We observe that from (6) that $x^{m-1} < 2y^{n-1}$ implying $x < 2^{\frac{1}{m-1}} y^{\frac{n-1}{m-1}}$. This together with (35), $d_3 \leq x - y < x$ and $2^\delta dd_2 \geq 2$ gives

$$y^n < 2^{\frac{2+3\epsilon}{m-1} - 1 - \epsilon} y^{2+2\epsilon + \frac{n-1}{m-1}(2+3\epsilon)}. \tag{36}$$

From (34), we obtain $m < 4+5\epsilon$ and further from (36), we get $n < 2+2\epsilon+\frac{n-1}{m-1}(2+3\epsilon)$ if $m > 3$.

Let $\epsilon = \frac{3}{4}$ and $N_\epsilon = 1$. Then $m \leq 7$ and further $7 \leq n \leq 17$ if $m = 6$ and $n \in \{8, 9\}$ if $m = 7$. Let $m = 7, n = m + 1 = 8$. Then $d_2 d_3 = 1$ and we get from the first inequality of (34) and $y < x$ that $x^m < x^{4+4\epsilon} = x^7$ implying $7 = m < 7$, a contradiction. Let $m = 7, n = m+2 = 9$. Then $d_2 d_3 \leq y+1$ and we get from (35) with $x < 2^{\frac{1}{m-1}} y^{\frac{n-1}{m-1}}$, $d_3(y-1) < y^2$ and $2^\delta dd_2 \geq 2$ that $y^n < 2^{\frac{2+2\epsilon}{m-1}-1-\epsilon} y^{2+3\epsilon+\frac{n-1}{m-1}(2+2\epsilon)} < y^9$ which is a contradiction again. Let $m = 6$ and $n \in \{11, 16\}$. From Nesterenko and Shorey [NeSh98], we get $y \leq 8, 15$ when $n = 11, 16$, respectively. For $2 \leq y \leq 15$ and $y + 1 \leq x \leq (\frac{y^n-1}{y-1}))^{\frac{1}{m-1}}$, we check that (6) does not hold. Therefore $n \notin \{11, 16\}$ when $m = 6$. Hence we have the first assertion of Theorem 5.

Now we take $\epsilon = \frac{1}{18}$. Since $m \leq 7$ and $G < x$, we get an explicit bound of x, y, m, n from (33) if $N < N_{\frac{1}{18}}$, implying Theorem 5 in that case. Thus we may suppose that $N \geq N_{\frac{1}{18}}$. Then we obtain from (34) with $\epsilon = \frac{1}{18}$ that $m < 4+5\epsilon$ implying $m \in \{3, 4\}$ and further from (36) that $n < 5$ if $m = 4$. This is a contradiction for $m = 4$ since $n > m$ and $n \in \mathbb{Z}$.

Let $m = 3$. We rewrite (6) as

$$(2x+1)^2 = 4(y^{n-1} + \cdots + y) + 1 \tag{37}$$

By [NeSh98], we may assume that $n \neq 5$. Let $n = 4$ and denote by $f(y)$ the polynomial on the right hand side of (37). Let $f'(\alpha) = 0$. Then $\alpha = \frac{-1\pm\sqrt{2}i}{3}$ and we check that $f(\alpha) \neq 0$. Therefore the roots of f are simple. Now we apply Baker [Bak69] to conclude that y and hence x are bounded by effectively computable absolute constant. Let $n \geq 6$. Now we rewrite (6) as

$$4y^n = (y-1)(2x+1)^2 + (3y+1). \tag{38}$$

Let G =gcd$(4y^n, (y-1)(2x+1)^2, 3y+1)$. Then $G = 4, 2, 1$ according as $4|(y-1), 4|(y-3)$ and $2|y$, respectively and we get from (38) that

$$\frac{4}{G}y^n = \frac{y-1}{G}(2x+1)^2 + \frac{3y+1}{G}. \tag{39}$$

Let

$$N = N(\frac{4y(y-1)(2x+1)(3y+1)}{G^3}) \leq \frac{y(y-1)(2x+1)(3y+1}{G} < \frac{6xy^3}{G_1}.$$

Let $\epsilon = \frac{1}{12}$. We obtain from Theorem 1 with $\epsilon = \frac{1}{12}$ that

$$\frac{4y^n}{G} < \begin{cases} N_{\frac{1}{12}}^{\frac{7}{4}} & \text{if } N < N_{\frac{1}{12}} \\ N^{1+\frac{1}{12}} & \text{if } N \geq N_{\frac{1}{12}}. \end{cases} \tag{40}$$

If $N < N_{\frac{1}{12}}$, then $y^n < N_{\frac{1}{12}}^{\frac{7}{4}}$ implying the assertion of Theorem 5. Hence we may suppose that $N \geq N_{\frac{1}{12}}$ and further y is sufficiently large. Then we have from $x^2 < 2y^{n-1}$ that

$$4y^n < (6\sqrt{2}y^{\frac{n+5}{2}})^{1+\frac{1}{12}}.$$

Therefore

$$n - \frac{13(n+5)}{24} < \frac{\frac{13}{12}\log(6\sqrt{2}) - \log 4}{\log y} < \frac{1}{24}$$

since y is sufficiently large. This is not possible since $n \geq 6$. Hence the assertion □

Acknowledgments

The second author would like to thank ISI, New Delhi where this work was initiated during his visit in July 2011.

References

[BEN10] M.A. Bennett, J.S. Ellenberg and N.C. Ng, *The Diophantine equation* $A^4 + 2^\delta B^2 = C^n$, International Journal of Number Theory, **6** (2010), no. 2, 311–338.

[Bak94] A. Baker, *Experiments on the abc-conjecture*, Publ. Math. Debrecen **65**(2004), 253–260.

[Bak69] A. Baker, *Bounds for the solutions of the hyperelliptic equation*, Proc. Cambridge Philos. Soc. **65** (1969), 439–444.

[Beu04] F. Beukers, *The Diophantine equation* $Ax^p + By^q = Cz^r$, Lectures held at Institut Henri Poincare, September 2004, *http : //www.math.uu.nl/people/beukers/Fermatlectures.pdf* Publ. Math. Debrecen **65**(2004), 253–260.

[BHM02] Y. Bugeaud, G. Hanrot and M. Mignotte, *Sur l'equation diophantienne* $\frac{x^n-1}{x-1} = y^q$, Proc. London Math. Soc. **84**(2002), 59–78.

[BuMi02] Y. Bugeaud and and M. Mignotte, *L' equation de Nagell-Ljunggren* $\frac{x^n-1}{x-1} = y^q$, Enseign. Math. **48**(2002), 147–168.

[BuMi07] Y. Bugeaud and P. Mihailescu, *On the Nagell-Ljunggren equation* $\frac{x^n-1}{x-1} = y^q$, Math. Scand. **101**(2007), 177–183.

[ChSi09] I. Chen and S. Siksek, *Perfect powers expressible as sums of two cubes*, Journal of Algebra **322** (2009), 638–656.

[Coh07] H. Cohen, *Number Theory, Volume II: Analytic and Modern Tools*, GTM 240, Springer-Verlag, 2007.

[DaGr95] H. Darmon and A. Granville, *On the equations $z^m = F(x, y)$ and $Ax^p + By^q = cZ^r$*, Bull. London Math. Soc. **27** (1995), 513–543.

[Dus99a] P. Dusart, *The k^{th} prime is greater than $k(\ln k + \ln \ln k - 1)$ for $k \geq 2$*, Math. Comp. **68** (1999), 411–415

[Dus99b] P. Dusart, *Inégalitiés explicites pour $\psi(X), \theta(X), \pi(X)$ et les nombres premiers*, C. R. Math. Rep. Acad. Sci. Canada **21(1)**(1999), 53-59.

[Elk91] N. Elkies, *ABC implies Mordell*, Int. Math. Res. Not. **7** (1991), [Ch 9].

[ErSe75] P. Erdős and J. L. Selfridge, *The product of consecutive integers is never a power*, Illinois J. Math. **19** (1975), 292-301.

[GrTu02] A. Granville and T. J. Tucker, *It's as easy as abc*, Notices of the AMS, **49**(2002), 1224-31.

[GyHaSa04] K. Győry, L. Hajdu and , N. Saradha, *On the Diophantine equation $n(n + d) \cdots (n + (k - 1)d) = by^\ell$*, Canad. Math. Bull. **47**(2004), 373–388.

[Kra99] A. Kraus, *On the Equation $x^p + y^q = z^r$: A Survey*, Ramanujan Journal **3** (1999), 315–333.

[Lai04] S. Laishram, *Topics in Diophantine equations*, M.Sc. Thesis, TIFR/Mumbai University, 2004, online at http: //www.isid.ac.in/∼shanta/MScThesis.pdf.

[LaSh04] S. Laishram and T. N. Shorey, *Number of prime divisors in a product of terms of an arithmetic progression*, Indag. Math., **15(4)** (2004), 505-521.

[Lju43] W. Ljunggren, *Noen Setninger om ubestemte likninger av formen $(x^n - 1)/(x - 1) = y^q$*, Norsk. Mat. Tidsskr. 1. Hefte **25**(1943), 17–20.

[Nag20] T. Nagell, *Note sur ℓ' équation indéterminée $(x^n - 1)/(x - 1) = y^q$*, Norsk. Mat. Tidsskr. **2** (1920), 75-78.

[NeSh98] Yu.V. Nesterenko and T. N. Shorey, *On an equation of Goormaghtigh*, Acta Arith. **83** (1998), 381–389.

[PSS07] B. Poonen, E.F. Schaefer and M. Stoll, *Twists of $X(7)$ and primitive solutions to $x^2 + y^3 = z^7$*, Duke Math. J. **137** (2007), 103–158.

[Rob55] H. Robbins, *A remark on Stirling's formula*, Amer. Math. Monthly **62** (1955), 26-29.

[Rob83] G. Robin, *Estimation de la fonction de Tchebychef θ sur le k-ieme nombre premier et grandes valeurs de la fonction $\omega(n)$ nombre de diviseurs premiers de n*, Acta Arith. **42** (1983), 367-389.

[Sar] N. Saradha, *Applications of Explicit abc-Conjecture on two Diophantine Equatiosn*, a preprint.

[SaSh05] N. Saradha and T. N. Shorey, *Contributions towards a conjecture of Erdos on perfect powers in arithmetic progressions*, Compositio Math., **141** (2005), 541-560.

[SaSh03] N. Saradha and T. N. Shorey, *Almost squares and factorisations in consecutive integers*, Compositio Math. **138** (2003), 113-124.

[Sho02a] T.N. Shorey, *Powers in arithmetic progression*, In: G. Wüstholz (ed.) A Panorama in Number Theory or The view from Baker's Garden, Cambridge Univ. Press (2002), 325-336.

[Sho02a] T. N. Shorey, *An equation of Goormaghtigh and diophantine approximations*, Current Trends in Number Theory, edited by S.D.Adhikari, S.A.Katre and B.Ramakrishnan, Hindustan Book Agency, New Delhi (2002), 185–197.

[Sho02b] T.N. Shorey, *Powers in arithmetic progression* (II), Analytic Number Theory, RIMS Kokyuroku (2002), Kyoto University.

[Sho99] T. N. Shorey, *Exponential diophantine equations involving products of consecutive integers and related equations*, Number Theory ed. R.P. Bambah, V.C. Dumir and R.J. Hans-Gill, Hindustan Book Agency (1999), 463-495.

[ShTi90] T. N. Shorey and R. Tijdeman, *Perfect powers in products of terms in an arithmetical progression*, Compositio Math. **75** (1990), 307-344.

[SiSt90] S. Siksek and M. Stoll, *Partial descent on hyperelliptic curves and the generalized Fermat equation $x^3 + y^4 + z65 = 0$*, Bulletin of the LMS, in press.

[TaWi95] R. Taylor and A. Wiles, *Ring-theoretic properties of certain Hecke algebras*, Annals of Mathematics **141** (1995), 553–572.

[Wil95] A. Wiles, *Modular elliptic curves and Fermat's Last Theorem*, Annals of Mathematics **141** (1995), 443–551.

[ABC3] *ABC triples*, page maintained by Bart de Smit at $http : //www.math.leidenuniv.nl/ \sim desmit/abc/index.php?sort = 1$, see also $http : //rekenmeemetabc.nl/Synthese\_resultaten$, $http : //www.math.unicaen.fr/ \sim nitaj/abc.html$.

E-mail address: shanta@isid.ac.in

Stat Math Unit, Indian Statistical Institute, 7 SJS Sansanwal Marg, New Delhi 110016, India

E-mail address: shorey@math.iitb.ac.in

Department of Mathematics, Indian Institute of Technology Bombay, Powai, Mumbai 400076, India